SENSITIVITY TO ERROR OF THE TRUNCATED HILBERT TRANSFORM TECHNIQUE FOR INTERIOR RECONSTRUCTION

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ABSTRACT

Historically, computed tomography reconstructions from truncated projection data have been considered non-unique. However, several recent results suggest that if the density of the object is known on some small region within the region of interest (ROI) then a unique and stable reconstruction of the complete ROI may be possible.

Unfortunately, prior knowledge of the exact density of an object being scanned is uncommon, and as such an experimentally determined estimate must be used in its place. This estimate will naturally contain errors. We have performed several reconstruction simulations to establish the sensitivity of the proposed algorithm to errors in the prior knowledge. Our results suggest that the most important element of the prior knowledge is its mean value, while perturbations of the details cause fewer problems in the reconstructed images.

***Index Terms*—** Computed tomography, incomplete data, sensitivity to error

1. INTRODUCTION

The majority of computed tomography reconstructions are performed by filtered backprojection (FBP) for which a complete set of line integral data are required. However, in a significant minority of imaging situations only incomplete data are available. We are particularly interested in what has become known as the "interior CT problem" where the sensing device is not large enough to measure line integrals along all rays passing through the object from the source and therefore the data sets for some or all projection angles are truncated [1]. For simplicity we assume that 1D data are collected only in a plane intersecting the object and that the detector is positioned such that a convex region (Region A in Fig. 1) on the object in the plane is observed for all of the projection angles. We also restrict our attention to parallel ray data; the extension to fan beam data is relatively straightforward [1].

The interior CT problem has been shown to be non-unique [1, 2]. Thus, even though line-integral data may be available

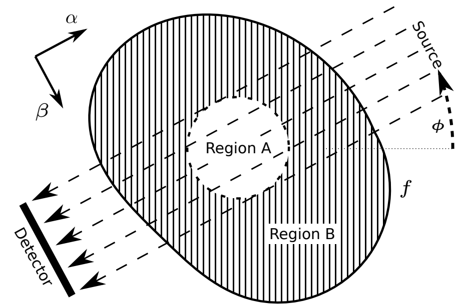

Fig. 1: Geometry for the scanning operation showing truncated line integral data gathered at angle ϕ and the corresponding regions A and B within the object.

for all angles for points in Region A, that does not allow a unique solution to be found when there exists another disjoint region (B) within the object for which line-integral data is only available over less than a full set of angles. It is therefore necessary to introduce some form of extra prior knowledge in order to achieve a useful and accurate solution. Such prior knowledge is also likely to be helpful to help overcome the inevitable error on the measured data.

Despite the somewhat negative tone of the previous paragraph, a series of recent results have suggested what nature of prior knowledge might enable a unique solution to be found for the interior CT problem. In particular, knowledge of a small part of Region A has been shown to enable a solution in principle [3]. A key part of the approach has been to reformulate the inverse Radon transform from the single-step FBP to a 2-step process: the 1D line integral data sets are first differentiated before backprojection, then the image so formed is Hilbert transformed along lines [4]. Following the notation of Clackdoyle and Defrise [1], the measured data are

$$p(\phi, s) = \int_{-\infty}^{\infty} f(r\vec{\alpha} + s\vec{\beta})\,\mathrm{d}r \quad \text{for } \phi \in (0, \pi),\ s \in (s_1, s_2), \tag{1}$$

where ϕ is the angle of the integral lines from the horizontal, $\vec{\alpha}$ and $\vec{\beta}$ are unit vectors oriented parallel and orthogonal to those lines, respectively, and f is the unknown density func-

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tion (see Fig. 1). Back projection of the differentiated line integral data sets can be represented as

$$q(\vec{x}) = \int_0^{\pi} p'(\phi, \vec{x} \cdot \vec{\beta}) \, \mathrm{d}\phi, \tag{2}$$

where the derivative of $p(\phi, s)$ is with respect to s. $q(\vec{x})$ is referred to as the differentiated back-projection (DBP). In the second step, an estimate $\hat{f}$ of the unknown density is formed by performing an inverse Hilbert transform along lines at a given fixed angle:

$$\hat{f}(r\vec{\alpha}+s\vec{\beta}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} q(r'\vec{\alpha}+s\vec{\beta}) \, h(r-r') \, |_{\phi=\phi_0} \, \mathrm{d}r', \tag{3}$$

where $h(r) = (\pi r)^{-1}$ and for some suitable angle ϕ_0. For this to work, however, the lines along which the Hilbert transform are performed must not encounter Region B [1]. Prior knowledge is required to overcome this limitation.

The key advantage of the 2-step process is that the differentiation operation in (2) is a local operation. As a consequence, the combination of differentiation and back projection operations does not cause smearing of the contents of region B into region A in the way that it occurs when conventional FPB is performed.

Very recently a number of authors have reported encouraging results from their simulation studies, seeking to demonstrate that the 2-step differentiation/Hilbert transform approach, coupled with prior knowledge in the form of a small region of ground truth, can lead to a usefully accurate reconstruction using projections onto convex sets (POCS) [5–8]. In this paper we revisit aspects of their work and seek to investigate the sensitivity of the reconstructions to various errors in the "known" data, henceforth referred to as the truth. Specifically, we investigate the sensitivity of the algorithm to constant offset errors of different magnitude, to the case where only the mean density of the truth is known, and to random corruptions of the truth.

2. RECONSTRUCTION METHOD

The reconstruction algorithm used in this paper is based on the DBP-POCS algorithm described by Kudo et al. in [5], which in turn is a modified version of the algorithm described by Defrise et al. in [8]. The core of the method is repeated here for convenience.

Step 1: Select the Hilbert lines. Lines should be chosen such that all lines intersect with the region of known density.

Step 2: Calculate the truncated Hilbert transforms. The Hilbert transform within Region A of the chosen lines can be calculated according to (2).

Step 3: Invert the truncated Hilbert transforms. The truncated Hilbert transform on each of the lines chosen in Step 1 can be inverted using the POCS method.

Within the algorithm the Hilbert lines are reconstructed independently, so the following description outlines the approach taken for a single line $f(t)$ of the image $f(\vec{x})$. The POCS method aims to find $\hat{f}(t)$ which belongs to the intersection of the following five sets:

$$\begin{aligned}
C_1 &= \left\{ \hat{f} \in L^2(\mathbb{R}) \,|\, \mathcal{H}\hat{f}(t) = q(t) \;\text{ for } t \in \Lambda_H \right\} \\
C_2 &= \left\{ \hat{f} \in L^2(\mathbb{R}) \,|\, \hat{f}(t) = 0 \;\text{ for } t \notin \Lambda \right\} \\
C_3 &= \left\{ \hat{f} \in L^2(\mathbb{R}) \,|\, \hat{f}(t) = f_K(t) \;\text{ for } t \in \Lambda_K \right\} \\
C_4 &= \left\{ \hat{f} \in L^2(\mathbb{R}) \,|\, \int_{\Lambda} \hat{f}(t) \, \mathrm{d}t = C_\Lambda \right\} \\
C_5 &= \left\{ \hat{f} \in L^2(\mathbb{R}) \,|\, \hat{f}(t) \geq 0 \;\text{ for } t \in \mathbb{R} \right\},
\end{aligned} \tag{4}$$

where Λ_H, Λ, and Λ_K denote the support of the truncated Hilbert transform, the full object, and the known region respectively, following the notation of [5]. $\mathcal{H}$ denotes the Hilbert transformer operator, $f_K(t)$ the density in the known region, and $q(t)$ the value of $q(\vec{x})$ from (2) on the line. Note that C_Λ is known to be the measured value of the projection along the line.

In each iteration of the algorithm, the estimate is projected onto each of the above sets, as

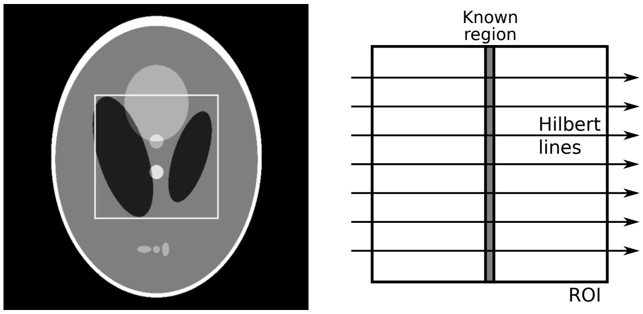
$$\hat{f}_0 = 0, \qquad \hat{f}_i = P_5 P_4 P_3 P_2 P_1 \hat{f}_{i-1}, \tag{5}$$

where $\hat{f}_i$ is the estimate of the solution after i iterations, and P_k is the operator for projecting onto set C_k . Operators P_2, P_3, and P_5 have straightforward expressions and details for P_1 and P_4 can be found in [5]. In short, P_1 forces the Hilbert transform of the line to match the DBP obtained from the measured data; P_2 enforces the support constraint; P_3 sets the density in the known region equal to the truth; P_4 forces the line integrals to match the measured data; and P_5 enforces positivity since the solution is a density function.

3. RESULTS

Simulated data were used to test the reconstruction algorithm. The data were obtained by simulating parallel beam projections at 1200 angles on [0,π), with detector pixel width equal to the width of the image pixels. The standard Shepp-Logan phantom [9] shown in Fig. 2a was used, sampled such that the region of interest (ROI) was of dimension 256 x 256 pixels.

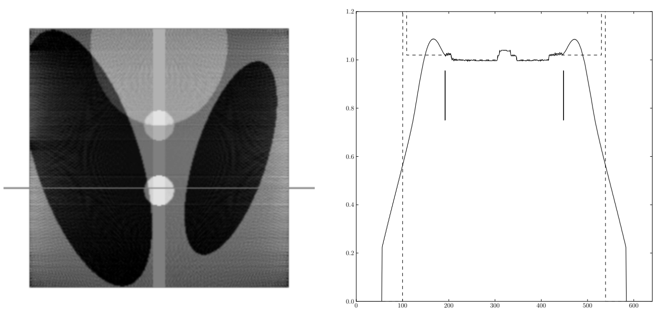
A truncated dataset was created by removing projection data corresponding to rays that did not pass through the ROI. As illustrated in Fig. 2b, the known region was a narrow strip down the centre of the ROI, twelve pixels wide. This configuration was selected to follow [5] and to allow the Hilbert lines to be chosen to be simply the horizontal lines of the image. The region of interest was chosen to be a square, rather than the more realistic shape of a circle, for the same reasons.



(a) Shepp-Logan phantom with the square region of interest overlaid in white. The window is [0.994,1.046].

(b) Configuration of the region of interest.

Fig. 2: The phantom and the region of interest.



(a) ROI image, with a line indicating the source of the profile plot. Window is [0.994, 1.046].

(b) Profile of reconstruction (solid) and of the original image (dashed). The solid vertical bars indicate the extent of the ROI.

Fig. 3: ROI reconstructed using the ground truth.

The algorithm was exercised with perfect data, then with three different forms of error in the truth to evaluate its sensitivity to each. Each reconstruction used 500 iterations of the POCS method.

Using error-free data in the known region produces a very good reconstruction as seen in Fig. 3a. The plot in Fig. 3b is a profile of the reconstruction along the line shown in Fig. 3a and shows the accuracy of the reconstruction.

Three trials were performed with a constant added to the truth, specifically -0.1, $+0.1$, and $+0.04$. Fig. 4 shows that a moderate constant error in the known region causes the reconstruction to suffer from negative cupping along the Hilbert lines. Also present, but less obvious in the image, is a negative constant error on the whole ROI. When the truth had a constant $+0.1$ error the reconstruction had similar but opposite artifacts – positive cupping on the Hilbert lines and a positive constant error on the whole ROI. Fig. 6 shows a profile of the reconstruction obtained when the known data had a smaller constant error. The positive cupping and positive constant offset can be seen clearly.

Kudo et al. [5] suggest that good reconstruction may be

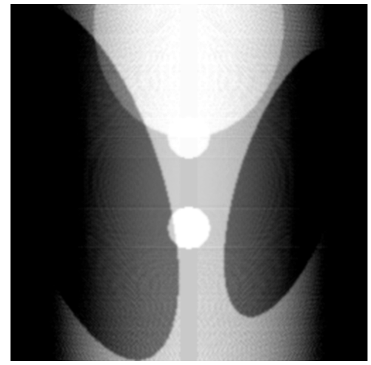


Fig. 4: Result due to a constant offset error of -0.1 in the known region. Window is [0.879, 0.931].

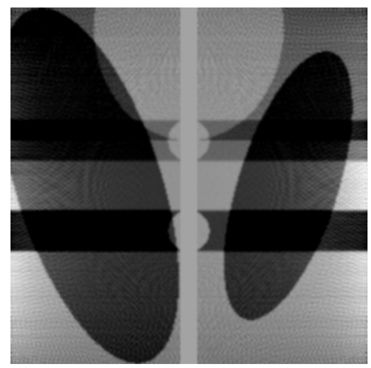


Fig. 5: Mean value of known region used as truth. Window is [0.994, 1.046].

obtainable with knowledge of only the mean density on some region. They present an example of a reconstruction obtained with the mean over the known region substituted for the true data, which appears to work well. However, the region they used for this was relatively featureless, and so the mean would not have been very different from the actual truth. Fig. 5 illustrates the outcome when the mean of the known region is applied instead of the truth when the region contains features. Clearly, attempting to use a mean value when the ground truth contains features does not produce a quality image.

The final form of error trialled was the presence of noise in the truth. The noise added was Gaussian with standard deviation 0.02, which is of the same order as the features in the ROI. Fig. 7 shows that the reconstruction contains considerably more distortion than the results of the other trials. However, it does not suffer from any cupping or constant shifts.

4. DISCUSSION AND CONCLUSION

The DBP/Hilbert algorithm was tested with different errors present in the truth. These errors can be broadly grouped into two categories: errors in the mean of the truth and errors in the shape of the truth. The effect of error in the mean of the

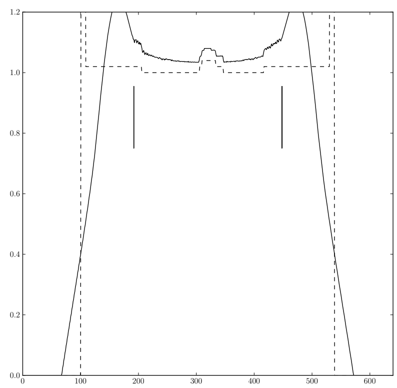


Fig. 6: Result due to a constant offset error of $+0.04$ in the known region.

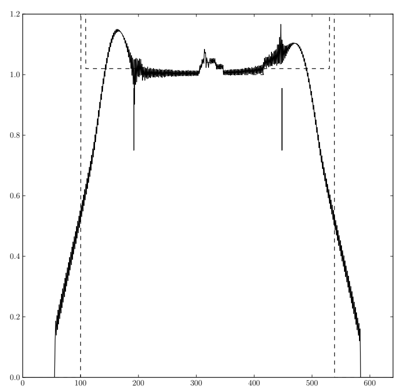


Fig. 7: Result when noise is present in the known region.

truth is to alter the mean of the reconstructed density as well as to introduce cupping artifacts. The effect of error in the shape of the truth is to distort the resulting reconstruction, as seen in Fig. 5 and Fig. 7.

Errors in the mean cause considerable harm to the reconstruction. This is because the main function of the known region in the algorithm is to determine the mean value of the resulting image. The features in the image are already reasonably well described by the Hilbert transform lines (obtained via DBP), so the particular shape of the density function in the known region is of less importance than its mean value. As long as the truth used in the algorithm has approximately the correct mean on each of the Hilbert lines, the algorithm should produce a good estimate of the original object.

The 4% offset error assumed in generating Fig. 6 is relatively large. In practice any scanning process used to obtain an estimate for the truth needs to be able to distinguish the features in the image. Since the features in this image are of the order of 2%, it is reasonable to expect the scanning process to have an accuracy better than 2% and thus obtaining a truth estimate with less than 4% error should be possible.

Fig. 5 suggests that knowing only the mean density across a region is insufficient to produce a good reconstruction. If the approximate mean density was known across a relatively homogeneous region, the liver for example, the prior knowledge may still be sufficient. If however the mean density across a heavily featured region was known, the results indicate that this would be insufficent to achieve a useful reconstruction with the algorithm tested.

In conclusion, the interior reconstruction method examined in this paper appears to be more sensitive to errors in the mean of the prior knowledge than to errors in the particular shape. Therefore, in order to obtain reliable reconstructions on an interior region of interest from truncated line integral data using this method, one must ensure that the density data applied as prior knowledge have a reliable mean.

5. REFERENCES

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